

# Lecture 5 Instructor Outline

## Hamilton's Principle and the Continuum Euler–Lagrange Equations

105 minutes — student notes are canonical; this is the board/teaching distillation

**Core arc:**

$$\delta \mathcal{S}_d = 0 \rightarrow \delta \mathcal{S} = 0 \rightarrow \text{integration by parts} \rightarrow \text{Euler–Lagrange}$$

$$\text{summation by parts} / A^T \longrightarrow \text{integration by parts} / \text{adjoint}$$

**Protected goal:** derive the continuum Euler–Lagrange equation directly from arbitrary fixed-endpoint path variations and make the boundary term visible. If time is short, compress shortest path/Fermat and the total-derivative note first.

**0–8 min: Start exactly where Lecture 4 stopped**

**BOARD**

$$\mathcal{S}_d = \sum_{n=0}^{N-1} L\left(x_n, \frac{x_{n+1} - x_n}{\Delta}\right) \Delta$$

$$\frac{\frac{\partial L_n}{\partial \dot{x}_n} - \frac{\partial L_{n-1}}{\partial \dot{x}_{n-1}}}{\Delta} - \frac{\partial L_n}{\partial x_n} = 0$$

**SAY** Last time, stationarity of one discrete sum gave one equation at every interior node. The first term is a discrete time derivative.

**BOARD**

$$\frac{\frac{\partial L_n}{\partial \dot{x}_n} - \frac{\partial L_{n-1}}{\partial \dot{x}_{n-1}}}{\Delta} \longrightarrow \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right)$$

so we expect

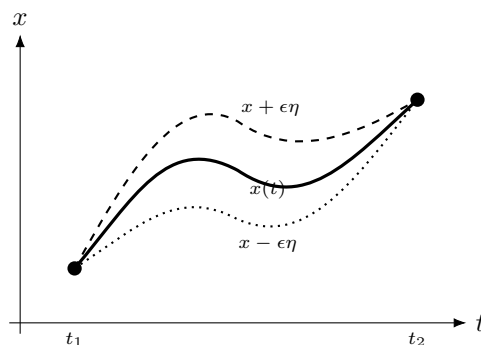
$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0.$$

**SAY** Today we do not assume this continuum equation. We derive it directly from a continuous path.

**ASK** What replaced “vary each interior node” when the nodes become infinitely dense? *Vary the function locally.*

**8–20 min: A path is a function — define a variation**

**BOARD** Draw/project neighboring fixed-endpoint paths.



## BOARD

$$x_\epsilon(t) = x(t) + \epsilon\eta(t)$$

Fixed endpoints:

$$\eta(t_1) = \eta(t_2) = 0.$$

Interior:  $\eta(t)$  arbitrary.

**SAY**  $\epsilon$  says how far we move in path space;  $\eta(t)$  gives the shape of that move. We vary an entire function, not one number.

## BOARD

$$\delta\mathcal{S}[x; \eta] \equiv \left. \frac{d}{d\epsilon} \mathcal{S}[x + \epsilon\eta] \right|_{\epsilon=0}$$

$$\delta x = \eta, \quad \delta \dot{x} = \frac{d}{dt}(\delta x) = \dot{\eta}.$$

**CHECK** Connect explicitly to Lecture 4: varying one  $x_n$  changed neighboring slopes; varying  $x(t)$  changes  $\dot{x}(t)$ .

## 20–34 min: Hamilton’s principle and the chain rule

### BOARD

$$\mathcal{S}[x] = \int_{t_1}^{t_2} L(x, \dot{x}, t) dt$$

$$\delta\mathcal{S} = 0, \quad \delta x(t_1) = \delta x(t_2) = 0.$$

**SAY** Stationary means first variation zero. Do not say “minimum action” as the general statement: stationary paths can be minima, maxima, or saddles in path space.

### BOARD

$$\mathcal{S}[x_\epsilon] = \int_{t_1}^{t_2} L(x + \epsilon\eta, \dot{x} + \epsilon\dot{\eta}, t) dt.$$

Differentiate with respect to  $\epsilon$ :

$$\delta\mathcal{S} = \int_{t_1}^{t_2} \left[ \frac{\partial L}{\partial x} \eta + \frac{\partial L}{\partial \dot{x}} \dot{\eta} \right] dt.$$

**SAY** Pause here. The path enters  $L$  in two ways: through its value  $x$ , and through its slope  $\dot{x}$ . This is the same chain-rule structure as the discrete calculation.

**ASK** Why can we not yet say the bracket multiplying  $\eta$  is zero? *Because the second term contains  $\dot{\eta}$ , not  $\eta$ .*

**STOP** Do not compress the next integration-by-parts step if students look uncertain. It is the conceptual center of the lecture.

## 34–50 min: Integration by parts moves the derivative

**BOARD** Start from the troublesome term:

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{x}} \dot{\eta} dt.$$

**BOARD** Integration by parts:

$$\begin{aligned} \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{x}} \dot{\eta} dt &= \left[ \frac{\partial L}{\partial \dot{x}} \eta \right]_{t_1}^{t_2} \\ &\quad - \int_{t_1}^{t_2} \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) \eta dt. \end{aligned}$$

**SAY** The derivative moves from the arbitrary variation onto  $\partial L / \partial \dot{x}$ . The price is a minus sign and a boundary term.

**BOARD** Fixed endpoints:

$$\eta(t_1) = \eta(t_2) = 0 \quad \Rightarrow \quad \left[ \frac{\partial L}{\partial \dot{x}} \eta \right]_{t_1}^{t_2} = 0.$$

Thus

$$\delta \mathcal{S} = \int_{t_1}^{t_2} \left[ \frac{\partial L}{\partial x} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) \right] \eta(t) dt.$$

**SAY** Do not throw the boundary term away conceptually. It vanishes here because we chose fixed endpoints. If an endpoint were free, the same term would tell us something about the boundary condition.

**CHECK** Repeat course refrain: *boundary terms carry information*.

## 50–61 min: Localize the variation — one integral becomes a differential equation

**BOARD** Define

$$G(t) = \frac{\partial L}{\partial x} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right).$$

Stationarity gives

$$\int_{t_1}^{t_2} G(t) \eta(t) dt = 0$$

for every smooth  $\eta$  vanishing at the endpoints.

**ASK** Could  $G(t_*) > 0$  at one interior point and the integral still vanish for every possible variation?

**SAY** No. Choose  $\eta$  supported in a small neighborhood of  $t_*$  with the same sign. Then the integral cannot vanish. Therefore the coefficient must vanish pointwise.

**BOARD**

$$G(t) = 0 \quad \text{for every interior } t$$

so

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0.$$

**SAY** This is the continuum version of requiring  $\partial \mathcal{S}_d / \partial x_n = 0$  for every interior node.

**CHECK** Name only after the logic is clear: *fundamental lemma of the calculus of variations*; equation: *Euler–Lagrange*.

## 61–72 min: Put discrete and continuum structures side by side

**BOARD**

| Discrete                       | Continuum                      |
|--------------------------------|--------------------------------|
| $x_n$                          | $x(t)$                         |
| $\delta x_n$                   | $\eta(t) = \delta x(t)$        |
| neighboring differences        | $\dot{x}$                      |
| sum                            | integral                       |
| summation by parts / transpose | integration by parts / adjoint |
| equation at every node         | equation at every time         |

**BOARD** Operator language:

$$A_t = \frac{d}{dt}, \quad \langle f, g \rangle = \int f g dt,$$

$$\langle f, A_t g \rangle = [fg]_{t_1}^{t_2} + \langle -A_t f, g \rangle.$$

Hence, formally,

$$A_t^T = -\frac{d}{dt}$$

with the boundary term handled separately.

**BOARD**

$$\begin{aligned}\delta\mathcal{S} &= \langle L_x, \delta x \rangle + \langle L_{\dot{x}}, A_t \delta x \rangle \\ &= \langle L_x + A_t^T L_{\dot{x}}, \delta x \rangle + \text{boundary}.\end{aligned}$$

**SAY** Same structural move as Lecture 2: a quantity living on differences is returned to the underlying nodes/function by the adjoint.

**OPTIONAL** If this operator language feels like too much after the main derivation, state only the boxed adjoint relation and move on.

## 72–84 min: Mechanics returns — free particle and conservative force

**BOARD** Free particle:

$$L = \frac{1}{2}m\dot{x}^2, \quad \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad \frac{\partial L}{\partial x} = 0.$$

$$\boxed{\frac{d}{dt}(m\dot{x}) = 0} \quad \Rightarrow \quad \boxed{\ddot{x} = 0} \quad (m = \text{const}).$$

**SAY** Three organizations of the same free motion:

$$K = 0 \quad \Longleftrightarrow \quad p = \text{constant} \quad \Longleftrightarrow \quad \delta\mathcal{S} = 0 \text{ for } L = \frac{1}{2}m\dot{x}^2.$$

**BOARD** Conservative particle:

$$\boxed{L = T - V = \frac{1}{2}m\dot{x}^2 - V(x)}$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad \frac{\partial L}{\partial x} = -V'(x),$$

so

$$\boxed{m\ddot{x} = -V'(x) = F(x)}.$$

**BOARD** Define generalized momentum:

$$\boxed{p \equiv \frac{\partial L}{\partial \dot{x}}} \quad \Rightarrow \quad \boxed{\dot{p} = \frac{\partial L}{\partial x}}.$$

**SAY** In Cartesian mechanics this is ordinary momentum. The definition survives when the coordinate is not Cartesian.

## 84–94 min: Shortest path — E–L is not specifically mechanics

**BOARD** For a plane curve  $x = x(y)$ :

$$\boxed{\ell[x] = \int_{y_1}^{y_2} \sqrt{1 + (x')^2} dy}, \quad x' = \frac{dx}{dy}.$$

Set

$$f(x, x', y) = \sqrt{1 + (x')^2}.$$

Then

$$\frac{d}{dy} \frac{\partial f}{\partial x'} - \frac{\partial f}{\partial x} = 0.$$

Since  $f_x = 0$ ,

$$\frac{d}{dy} \left( \frac{x'}{\sqrt{1 + (x')^2}} \right) = 0 \Rightarrow x' = \text{constant}$$

and therefore

$$\boxed{x(y) = ay + b.}$$

**SAY** Euler–Lagrange is a statement about stationary functionals. Mechanics is one application.

**OPTIONAL** Fermat:

$$T[x] = \int_{y_1}^{y_2} \frac{\sqrt{1 + (x')^2}}{u(x, y)} dy, \quad \boxed{\delta T = 0.}$$

Or  $T \propto \int n(x, y) \sqrt{1 + (x')^2} dy$ .

**STOP** If behind schedule, skip the algebra after writing the arc-length functional and state that E–L gives a straight line.

## 94–101 min: Many coordinates

**BOARD**

$$\mathbf{q}(t) = (q_1, \dots, q_N), \quad \boxed{\mathcal{S}[\mathbf{q}] = \int L(\mathbf{q}, \dot{\mathbf{q}}, t) dt.}$$

$$\delta \mathcal{S} = \sum_{i=1}^N \int \left[ \frac{\partial L}{\partial q_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) \right] \eta_i dt.$$

Independent  $\eta_i$  imply

$$\boxed{\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, \dots, N.}$$

Vector form:

$$\boxed{\frac{d}{dt} \nabla_{\dot{\mathbf{q}}} L - \nabla_{\mathbf{q}} L = \mathbf{0}.}$$

Conjugate momenta:

$$\boxed{p_i = \frac{\partial L}{\partial \dot{q}_i}.}$$

**SAY** Nothing in the derivation required the  $q_i$  to be Cartesian positions.

## 101–105 min: Why generalized coordinates come next

**BOARD** Particle constrained to a circle:

$$x^2 + y^2 = R^2.$$

Instead of  $(x, y)$  plus a constraint force, use

$$\boxed{x = R \cos \theta, \quad y = R \sin \theta.}$$

Then

$$\dot{x}^2 + \dot{y}^2 = R^2 \dot{\theta}^2, \quad T = \frac{1}{2} m R^2 \dot{\theta}^2.$$

**SAY** The variational formalism is already prepared to use  $\theta$  directly. Choose coordinates that describe only the configurations the system is actually allowed to occupy.

**NEXT** Generalized coordinates and constraints.

**OPTIONAL** If 2–3 minutes remain, add the total-derivative freedom:

$$L' = L + \frac{dF(q, t)}{dt}$$

$$\mathcal{S}' = \mathcal{S} + F(q_2, t_2) - F(q_1, t_1)$$

so for fixed endpoints

$$\boxed{\delta\mathcal{S}' = \delta\mathcal{S}.}$$

**SAY** The Lagrangian is not unique; adding a total derivative changes only the boundary contribution.

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**End-of-lecture board summary**

$$\boxed{\delta\mathcal{S} = \int (L_x \delta x + L_{\dot{x}} \delta \dot{x}) dt}$$

$$\boxed{\text{integration by parts} \Rightarrow \delta\mathcal{S} = \int \left[ L_x - \frac{d}{dt} L_{\dot{x}} \right] \delta x dt + \text{boundary}}$$

$$\boxed{\delta x \text{ arbitrary in the interior} \Rightarrow \frac{d}{dt} L_{\dot{x}} - L_x = 0.}$$