

Lecture 4 Instructor Outline

Discrete Dynamics and the Action Principle

105 minutes — student notes are canonical; this is the board/teaching distillation

Core arc:

ordered configurations $\rightarrow$ clock $\rightarrow$ Newtonian update $\rightarrow$ stationary discrete action

Newton: initial-value march

$\longleftrightarrow$

Hamilton: fixed-boundary path comparison

Protected goal: derive the discrete Euler–Lagrange equation from a three-event picture before specializing to $L = T - V$.

0–7 min: Broad Newtonian law, then strip away the clock

BOARD

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}, \quad \mathbf{p} = M\dot{\mathbf{x}}$$

For constant M :

$$M\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, t).$$

SAY This is the broad mechanical law. Velocity dependence and explicit t -dependence often enter when we describe a reduced/driven subsystem. Fundamental laws need not pick out a preferred origin of time; we will return to that in Noether’s theorem.

SAY Now remove the clock completely. Keep only an ordered sequence of configurations.

BOARD

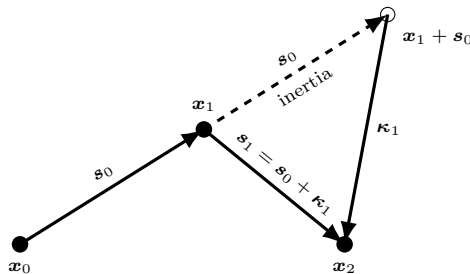
$$\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \dots, \quad \mathbf{s}_n = \mathbf{x}_{n+1} - \mathbf{x}_n, \quad \boldsymbol{\kappa}_{n+1} = \mathbf{s}_{n+1} - \mathbf{s}_n.$$

SAY $\mathbf{s}$ is a step, not yet a velocity. $\boldsymbol{\kappa}$ is a turn, not yet an acceleration. “Turn” means any change of the step: parallel $\boldsymbol{\kappa}$ lengthens it, antiparallel $\boldsymbol{\kappa}$ shortens it, transverse $\boldsymbol{\kappa}$ changes direction.

ASK What information has been used so far? *Only ordering and vector differences; no seconds, mass, or force.*

7–20 min: Move, then turn — discrete Newton geometry

BOARD Draw the configuration-space picture below (or project it briefly).



BOARD

$$\begin{aligned} \mathbf{x}_{n+1} &= \mathbf{x}_n + \mathbf{s}_n, \\ \mathbf{s}_{n+1} &= \mathbf{s}_n + \mathbf{K}(\mathbf{x}_{n+1}, \mathbf{s}_n, t_n). \end{aligned}$$

For the clean conservative case:

$$\mathbf{K} = \mathbf{K}(\mathbf{x}_{n+1}).$$

SAY Move using the present step; at the new configuration determine how the next step changes. This is semi-explicit: the second line uses the already-updated position, but no inverse solve is needed.

BOARD

$$S_n = (\mathbf{x}_n, \mathbf{s}_n), \quad S_{n+1} = F(S_n).$$

SAY This is the Lecture 1 state-update idea in mechanical clothing.

BOARD Accumulation identities:

$$\mathbf{s}_n = \mathbf{s}_0 + \sum_{j=1}^n \boldsymbol{\kappa}_j$$

$$\mathbf{x}_n = \mathbf{x}_0 + n\mathbf{s}_0 + \sum_{j=1}^{n-1} (n-j)\boldsymbol{\kappa}_j$$

SAY These are the discrete analogs of integrating acceleration once to get velocity and twice to get position.

CHECK For $K = K_0$, students should be able to see immediately

$$x_n = x_0 + ns_0 + \frac{n(n-1)}{2}K_0.$$

20–34 min: Three turn laws, then add the clock

BOARD 1. Free:

$$K = 0 \Rightarrow s_n = s_0 \Rightarrow \boxed{x_n = x_0 + ns_0}.$$

SAY Newton I already exists as a statement about an ordered sequence.

BOARD 2. Constant turn:

$$K = K_0 \Rightarrow \boxed{x_n = x_0 + ns_0 + \frac{n(n-1)}{2}K_0}.$$

BOARD 3. Spring-like turn:

$$\begin{aligned} K &= -cx, & x_{n+1} &= x_n + s_n, \\ & & s_{n+1} &= s_n - cx_{n+1}. \end{aligned}$$
$$\underbrace{\begin{pmatrix} x_{n+1} \\ s_{n+1} \end{pmatrix}}_{\Phi_c} = \begin{pmatrix} 1 & 1 \\ -c & 1-c \end{pmatrix} \begin{pmatrix} x_n \\ s_n \end{pmatrix}, \quad \boxed{\det \Phi_c = 1}.$$

SAY Only interpret this today as phase-space area preservation. Do not discuss modified energy.

BOARD Now add equal clock spacing $t_n = t_0 + n\Delta$:

$$\boxed{\mathbf{v}_n \approx \frac{\mathbf{s}_n}{\Delta}}, \quad \boxed{\mathbf{a}_{n+1} \approx \frac{\boldsymbol{\kappa}_{n+1}}{\Delta^2}}.$$

Then dynamics adds mass:

$$\boxed{M\mathbf{a} = \mathbf{F}} \Rightarrow \boxed{\boldsymbol{\kappa} \approx \Delta^2 M^{-1} \mathbf{F}}.$$

SAY Sequence geometry + clock + force/mass law are three different layers.

BOARD Difference language:

$$\mathbf{x} \xrightarrow{A_t} \mathbf{s} \xrightarrow{A_t} \boldsymbol{\kappa}, \quad D_t = \frac{1}{\Delta} A_t, \quad \boxed{D_t M D_t \mathbf{x} = \mathbf{F}}.$$

CHECK Gravity:

$$M\mathbf{a} = M\mathbf{g} \Rightarrow \mathbf{a} = \mathbf{g}, \quad \boldsymbol{\kappa} = \mathbf{g}\Delta^2.$$

Spring:

$$ma = -kx \Rightarrow a = -\frac{k}{m}x, \quad \boxed{c = \frac{k}{m}\Delta^2 = (\omega\Delta)^2}, \quad \omega = \sqrt{k/m}.$$

34–48 min: MATLAB demo — same update, three laws

BOARD / PROJECT

```
N = 40;
x = zeros(1,N+1); s = zeros(1,N+1);
x(1) = 0; s(1) = 1;
c = 0.10;

for n = 1:N
    x(n+1) = x(n) + s(n);    % move

    % K = 0;                % free
    % K = -1;               % constant turn
    K = -c*x(n+1);          % spring-like

    s(n+1) = s(n) + K;      % turn
end

figure; plot(0:N,x,'o-')
figure; plot(x,s,'o-'); axis equal
```

ASK What changes between the three simulations? *Only the rule for the turn.*

BOARD Physical spring parameters:

```
dt = 0.10; m = 2.0; k = 8.0;
c = (k/m)*dt^2;
```

SAY The dimensionless sequence parameter combines system physics and the clock.

STOP If the demo runs long, do not derive the oscillator solution $x_n = A \cos n\theta + B \sin n\theta$ on the board. The determinant-one map and the plots are enough.

48–62 min: From stationary points to stationary paths

SAY Before varying a whole path, recall the simpler equilibrium idea: stationary does not mean minimum.

BOARD

$$\boxed{\frac{dV}{dx} = 0}, \quad \boxed{\delta V = \frac{dV}{dx} \delta x}.$$

Sketch a one-dimensional potential with a minimum and maximum.

SAY If $\delta V = 0$ for every allowed δx , then $dV/dx = 0$. With many coordinates,

BOARD

$$\boxed{\delta V = (\nabla V)^T \delta \mathbf{x}}, \quad \boxed{\delta V = 0 \forall \delta \mathbf{x} \Rightarrow \nabla V = 0}.$$

Sketch the saddle $V(x, y) = x^2 - y^2$.

CHECK Use the word *stationary*: maxima, minima, and saddles all satisfy the first-variation condition.

SAY Now generalize again: instead of varying a point, vary an entire curve between fixed endpoints.

BOARD Shortest path, parameterized as $x = x(y)$:

$$\boxed{\ell[x] = \int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy, \quad \delta \ell = 0}.$$

BOARD Fermat / stationary travel time:

$$\boxed{T[x] = \int_{y_1}^{y_2} \frac{\sqrt{1 + (dx/dy)^2}}{u(x, y)} dy, \quad \delta T = 0}.$$

Optionally note $u = c_0/n$.

BOARD General pattern:

$$\boxed{\mathcal{J}[x] = \int f(x, x', y) dy, \quad x' = \frac{dx}{dy}, \quad \delta \mathcal{J} = 0}.$$

SAY This is the bridge: ordinary stationarity under every coordinate variation gives equations such as $\nabla V = 0$. Functional stationarity under every local path variation gives a differential equation at every point of the path.

BOARD Mechanics: $y \rightarrow t$, $x' \rightarrow \dot{x}$, $f \rightarrow L$:

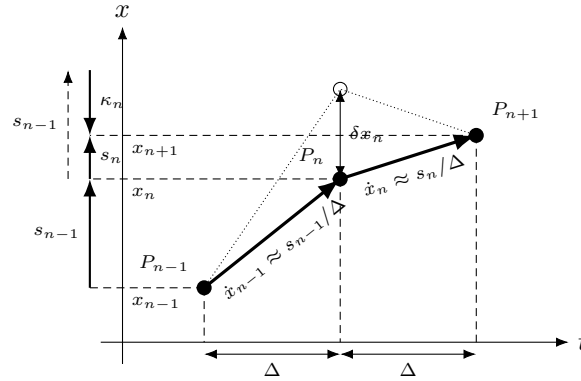
$$\boxed{L(x, \dot{x})}, \quad \boxed{\mathcal{S}[x] = \int_{t_0}^{t_1} L(x, \dot{x}) dt}, \quad \boxed{\delta \mathcal{S} = 0}.$$

SAY Do *not* assume $L = T - V$ yet. First discover what stationarity of a generic L requires.

ASK What has changed from Newton's initial-value picture? *We now compare whole paths between fixed boundaries rather than marching one state forward.*

62–74 min: Space-time geometry — events, slopes, and one-node variation

BOARD Draw/project the three-event picture.



SAY Vertical differences s, κ live in configuration space. Horizontal differences are clock intervals. The new variable $\dot{x}$ belongs to the combined space-time picture: it is the slope connecting two events.

BOARD

$$\dot{x}_{n-1} \approx \frac{x_n - x_{n-1}}{\Delta}, \quad \dot{x}_n \approx \frac{x_{n+1} - x_n}{\Delta}.$$

ASK If I move only the middle event vertically, what changes? *Both adjacent slopes, plus the explicit position argument of the outgoing L .*

74–89 min: Localize the action and derive discrete Euler–Lagrange

BOARD

$$\boxed{\mathcal{S}_d[x_0, \dots, x_N] = \sum_{j=0}^{N-1} L\left(x_j, \frac{x_{j+1} - x_j}{\Delta}\right) \Delta}$$

Endpoints fixed: x_0, x_N . Interior nodes vary.

SAY Although the action is a sum over the whole path, one interior node x_n appears in only two terms. This is true no matter how many points the path contains; three points are the minimum picture.

BOARD For variation of x_n , ignore every term independent of x_n and put the n -term first:

$$\boxed{\mathcal{S}_d(x_n) = L\left(x_n, \frac{x_{n+1} - x_n}{\Delta}\right) \Delta + L\left(x_{n-1}, \frac{x_n - x_{n-1}}{\Delta}\right) \Delta.}$$

Define the two discrete slopes and the corresponding values of L :

$$\dot{x}_n(x_n) = \frac{x_{n+1} - x_n}{\Delta}, \quad \dot{x}_{n-1}(x_n) = \frac{x_n - x_{n-1}}{\Delta},$$

$$L_n = L(x_n, \dot{x}_n), \quad L_{n-1} = L(x_{n-1}, \dot{x}_{n-1}).$$

Therefore

$$\boxed{\mathcal{S}_d(x_n) = \left[L_n(x_n, \dot{x}_n(x_n)) + L_{n-1}(x_{n-1}, \dot{x}_{n-1}(x_n)) \right] \Delta.}$$

SAY Now the chain rule is visible. The first term depends on x_n both explicitly and through $\dot{x}_n$. The previous term depends on x_n only through $\dot{x}_{n-1}$.

BOARD Chain rule:

$$\frac{\partial \mathcal{S}_d(x_n)}{\partial x_n} = \left[\frac{\partial L_n}{\partial x_n} + \frac{\partial L_n}{\partial \dot{x}_n} \frac{\partial \dot{x}_n}{\partial x_n} + \frac{\partial L_{n-1}}{\partial \dot{x}_{n-1}} \frac{\partial \dot{x}_{n-1}}{\partial x_n} \right] \Delta.$$

Compute the two slope derivatives explicitly:

$$\frac{\partial \dot{x}_n}{\partial x_n} = \frac{\partial}{\partial x_n} \left(\frac{x_{n+1} - x_n}{\Delta} \right) = -\frac{1}{\Delta},$$

$$\frac{\partial \dot{x}_{n-1}}{\partial x_n} = \frac{\partial}{\partial x_n} \left(\frac{x_n - x_{n-1}}{\Delta} \right) = +\frac{1}{\Delta}.$$

Hence

$$\frac{\partial \mathcal{S}_d(x_n)}{\partial x_n} = \Delta \frac{\partial L_n}{\partial x_n} - \frac{\partial L_n}{\partial \dot{x}_n} + \frac{\partial L_{n-1}}{\partial \dot{x}_{n-1}}.$$

BOARD Variation at the node:

$$\delta \mathcal{S}_d = \frac{\partial \mathcal{S}_d}{\partial x_n} \delta x_n = - \left[\frac{\frac{\partial L_n}{\partial \dot{x}_n} - \frac{\partial L_{n-1}}{\partial \dot{x}_{n-1}}}{\Delta} - \frac{\partial L_n}{\partial x_n} \right] \Delta \delta x_n = 0.$$

SAY The spacing Δ is nonzero, and the interior variation δx_n is arbitrary. Therefore the coefficient of δx_n must vanish at every interior node:

$$\boxed{\frac{\frac{\partial L_n}{\partial \dot{x}_n} - \frac{\partial L_{n-1}}{\partial \dot{x}_{n-1}}}{\Delta} - \frac{\partial L_n}{\partial x_n} = 0, \quad n = 1, \dots, N-1.}$$

SAY The first term is the discrete time derivative of $\partial L / \partial \dot{x}$. In the continuum limit,

$$\frac{\frac{\partial L_n}{\partial \dot{x}_n} - \frac{\partial L_{n-1}}{\partial \dot{x}_{n-1}}}{\Delta} \longrightarrow \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right),$$

so stationarity at every node becomes

$$\boxed{\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0.}$$

SAY This is the localization step: stationarity of one global sum becomes a local equation at every interior node. The explicit x_n -dependence gives $\partial L / \partial x$; changing the two adjacent slopes gives the time difference of $\partial L / \partial \dot{x}$.

CHECK Students should be able to point to the two affected edges in the picture and explain the signs $-1/\Delta$ and $+1/\Delta$. If they cannot, pause here rather than push ahead.

89–97 min: Generalized momentum and the free particle

BOARD The equation suggests

$$p_n = \frac{\partial L_n}{\partial \dot{x}_n}, \quad \frac{p_n - p_{n-1}}{\Delta} = \frac{\partial L_n}{\partial x_n}.$$

SAY This already looks like $\dot{p} = F$. Now ask which L makes the generalized momentum ordinary momentum.

BOARD Free particle:

$$L = \frac{1}{2} m \dot{x}^2 \quad \Rightarrow \quad p = \frac{\partial L}{\partial \dot{x}} = m \dot{x}.$$

Since $\partial L / \partial x = 0$,

$$p_n = p_{n-1} \Rightarrow \dot{x}_n = \dot{x}_{n-1} \Rightarrow \boxed{x_{n+1} - 2x_n + x_{n-1} = 0}.$$

SAY Newton I has reappeared, now from stationary action rather than from the state-update rule.

97–104 min: Discover $L = T - V$ and recover Newton

BOARD Start from Newton:

$$\boxed{\frac{d}{dt}(m\dot{x}) = F(x) = -\frac{dV}{dx}}.$$

With

$$T(\dot{x}) = \frac{1}{2} m \dot{x}^2, \quad m\dot{x} = \frac{\partial T}{\partial \dot{x}},$$

and, for this simple Cartesian case,

$$\frac{\partial T}{\partial x} = 0, \quad \frac{\partial V}{\partial \dot{x}} = 0,$$

Newton becomes

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}} \right) + \frac{\partial V}{\partial x} = 0,$$

so

$$\boxed{\left(\frac{d}{dt} \frac{\partial}{\partial \dot{x}} - \frac{\partial}{\partial x} \right) (T - V) = 0}.$$

SAY This is the inverse direction from the usual textbook presentation: Newton itself can be repackaged as the Euler–Lagrange operator acting on $T - V$. It gives a concrete reason to look at the difference $T - V$ before we appeal to the action principle.

BOARD Same discovery from the discrete E–L equation:

$$\dot{p} = F(x), \quad F(x) = -V'(x).$$

To get $p = m\dot{x}$:

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad \Rightarrow \quad T = \frac{1}{2} m \dot{x}^2.$$

To get $\partial L / \partial x = F = -V'$:

$$L \supset -V(x).$$

Therefore

$$\boxed{L = T - V = \frac{1}{2} m \dot{x}^2 - V(x)}.$$

SAY The minus sign is not guessed; it is what makes $\partial_x L = -V' = F$. Keep the scope explicit: this is the simple conservative Cartesian case, not the most general possible Lagrangian.

BOARD Contrast with Lecture 3:

$$\boxed{E = T + V} \quad \text{vs} \quad \boxed{L = T - V}.$$

BOARD Discrete Euler–Lagrange now gives

$$m \frac{x_{n+1} - 2x_n + x_{n-1}}{\Delta^2} = -V'(x_n).$$

With

$$s_{n-1} = x_n - x_{n-1}, \quad s_n = x_{n+1} - x_n,$$

this becomes

$$s_n = s_{n-1} - \frac{\Delta^2}{m} V'(x_n).$$

Together with

$$x_n = x_{n-1} + s_{n-1},$$

we are back to the opening rule: **move, then turn**.

SAY This closes the circle: the local Newtonian update and stationary discrete action generate the same discrete dynamics.

104–105 min: Close and preview Lecture 5

BOARD

global stationary sum $\implies$ local difference equation at every interior node

BOARD Preview only:

$$\mathcal{S}_d \longrightarrow \mathcal{S}[x] = \int L(x, \dot{x}, t) dt,$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0.$$

NEXT Lecture 5: continuum limit, integration by parts, Hamilton’s principle, boundary terms, and generalized coordinates/constraints as appropriate.

STOP Do not derive the continuum Euler–Lagrange equation today unless the class is substantially ahead. The discrete derivation is the conceptual payoff of Lecture 4.

If time slips: preserve the three-event variation and the discrete Euler–Lagrange derivation. Compress the matrix-map algebra and MATLAB discussion first.